Designing Optimal Voting Rules

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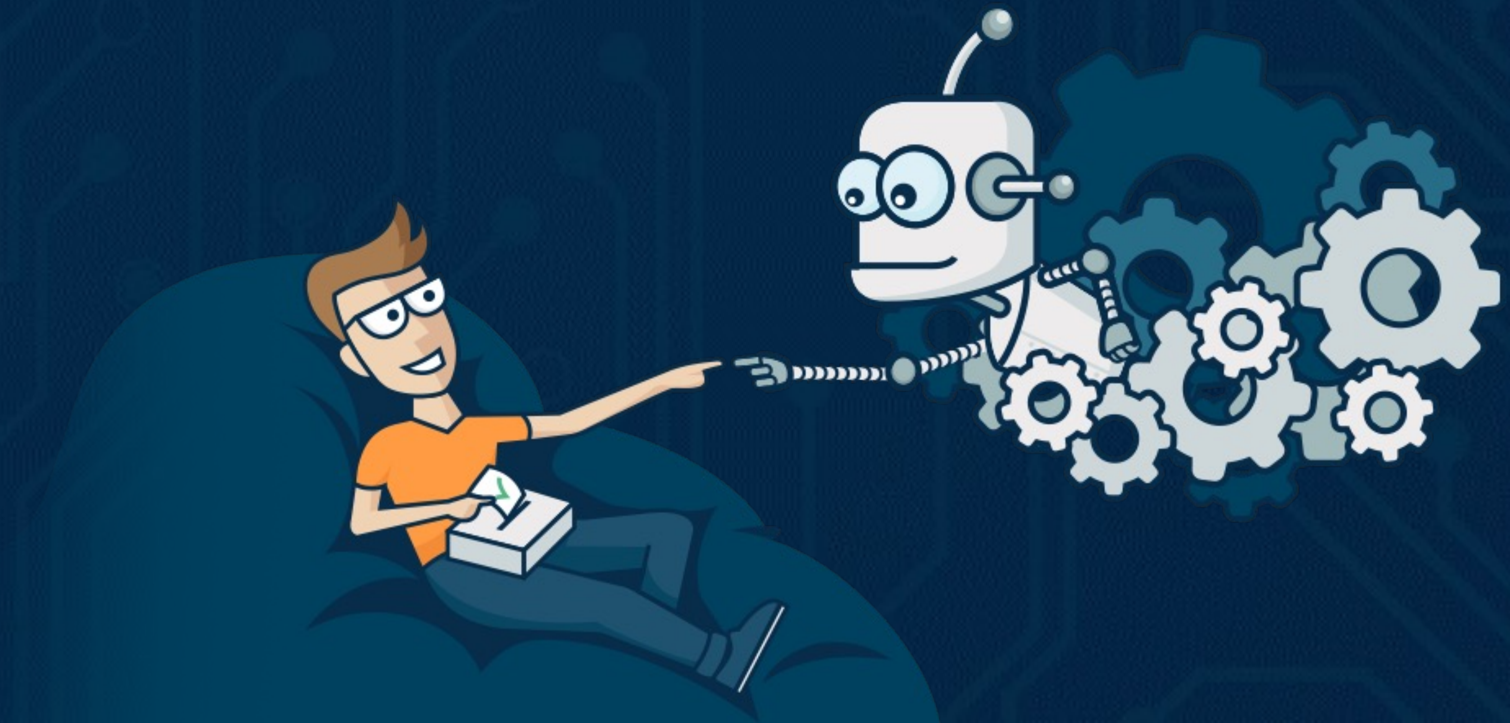


Collaborators







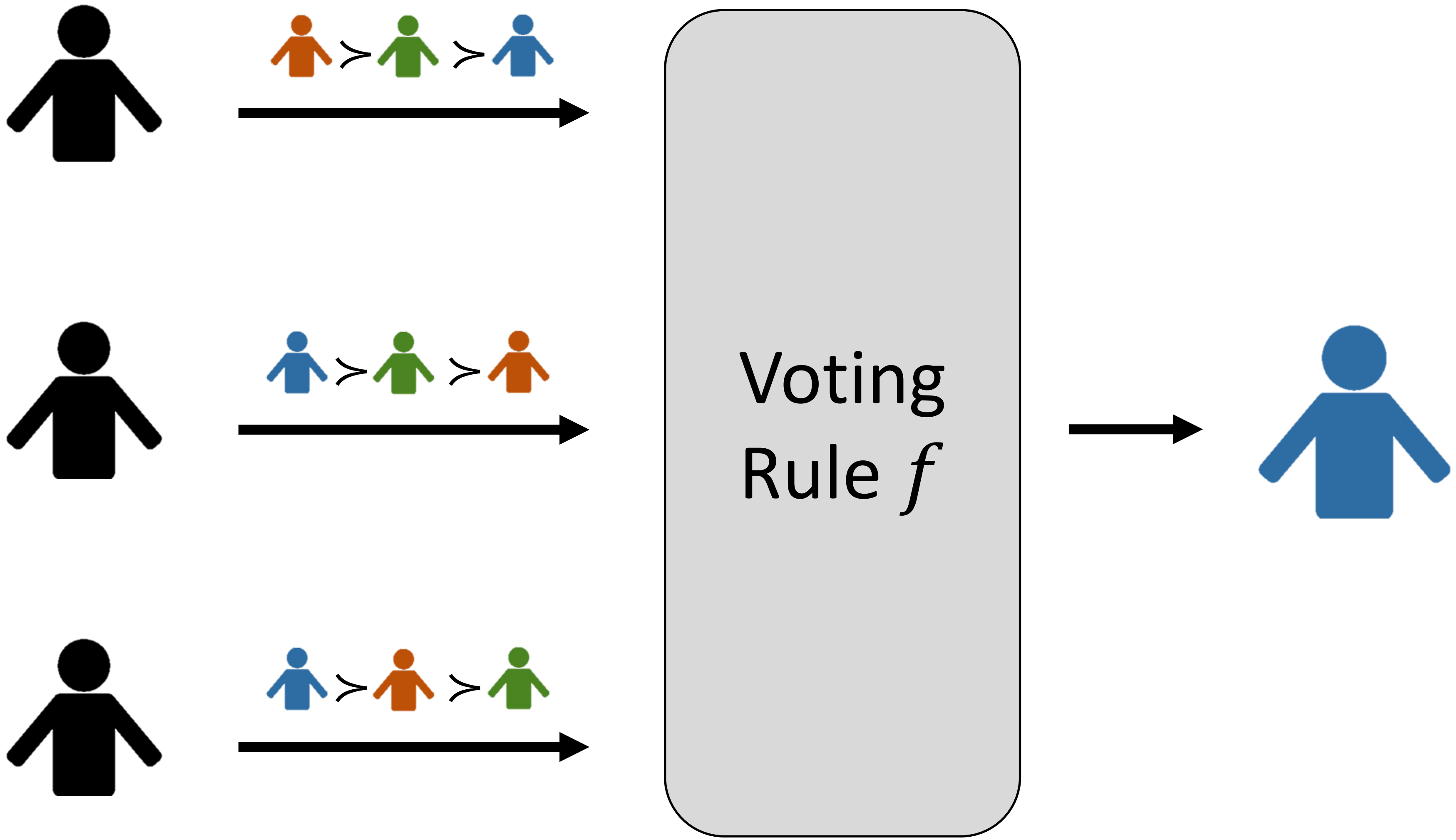


Voting

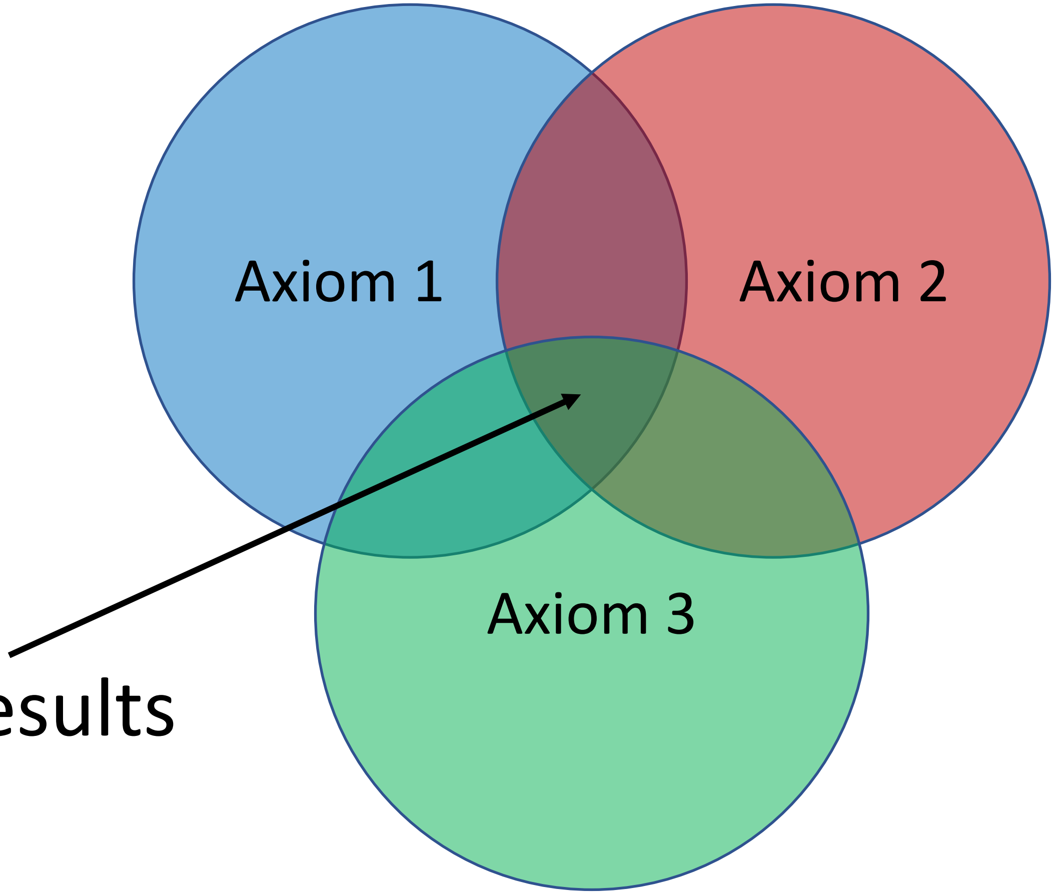
Algorithm for aggregating individual
preferences to make collective decisions

Applications of Voting

Voting with Ranked Ballots



Axiomatic Framework

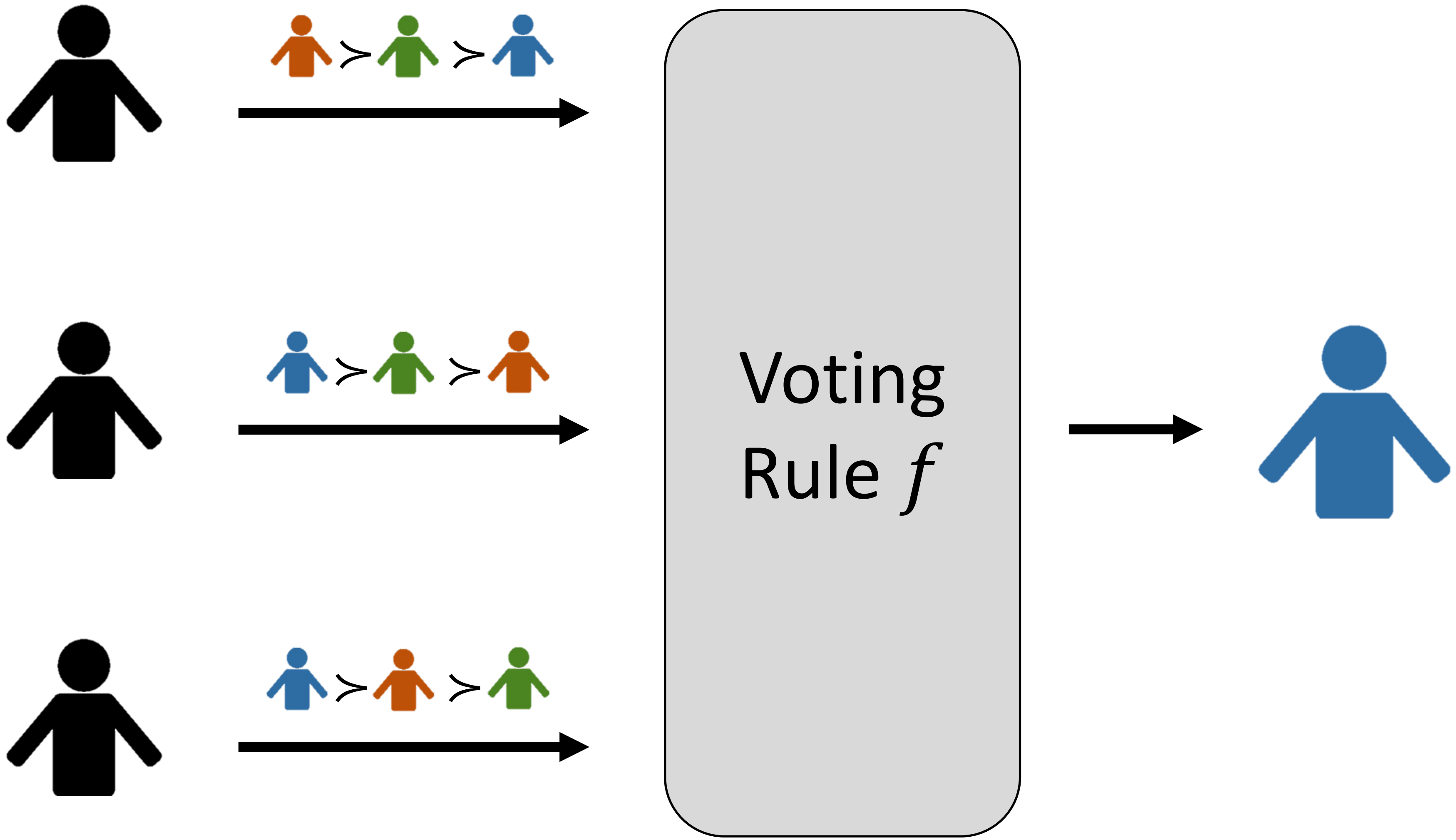


Impossibility Results

Axiomatic Framework

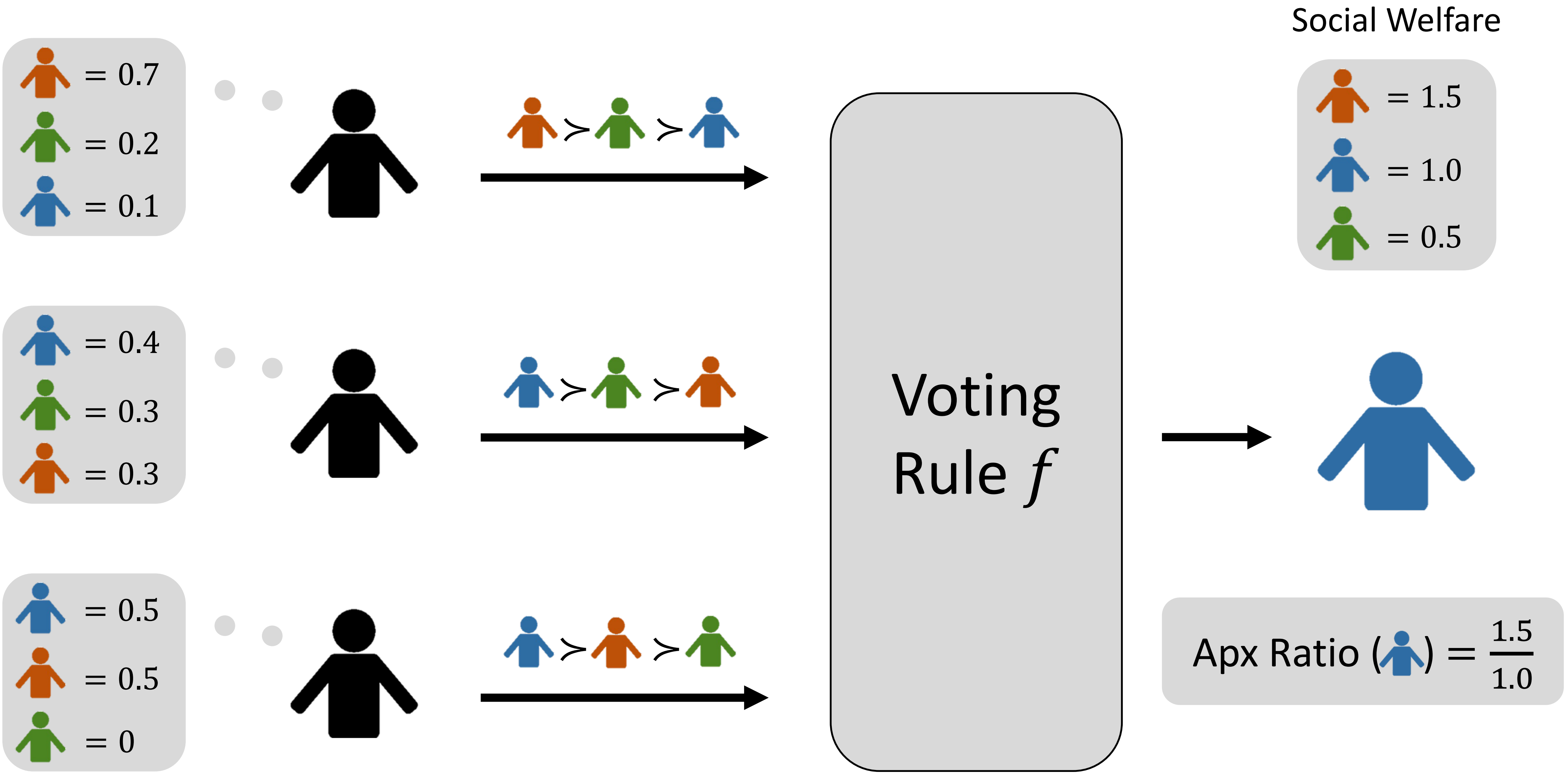
Sort: Criterion / Method	Majority	Maj. loser	Mutual maj.	Condorcet	Cond. loser	Smith/ ISDA	LIIA	IIA	Cloneproof	Monotone	Consistency	Participation	Reversal symmetry	Polytime/ resolvable		Summable	Later-no- Harm	Later-no- Help	No favorite betrayal	Ballot type	Ranks =	Ranks >2
Approval	Rated[a]	No	No	No[b][c]	No	No[b]	Yes	Yes[d]	Yes[e]	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes[f]	Yes	Approvals	Yes	No
Borda count	No	Yes	No	No[b]	Yes	No	No	No	Teams	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes	No	Ranking	No	Yes
Bucklin	Yes	Yes	Yes	No	No	No	No	No	No	Yes	No	No	No	O(N)	Yes	O(N)	No	Yes	If equal preferences	Ranking	Yes	Yes
Copeland	Yes	Yes	Yes	Yes	Yes	Yes	No	No[b]	Teams, crowds	Yes	No[b]	No[b]	Yes	$O(N^2)$	No	$O(N^2)$	No[b]	No	No[b]	Ranking	Yes	Yes
IRV (AV)	Yes	Yes	Yes	No[b]	Yes	No[b]	No	No	Yes	No	No	No	No	$O(N^2)$	Yes[g]	O(N!)[h]	Yes	Yes	No	Ranking	No	Yes
Kemeny–Young	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No[b]	Spoilers	Yes	No[b] [i]	No[b]	Yes	O(N!)	Yes	$O(N^2)$[j]	No[b]	No	No[b]	Ranking	Yes	Yes
Majority judgment[k]	Rated[l]	Yes[m]	No[n]	No[b][c]	No	No[b]	Yes	Yes[d]	Yes	Yes	No[o]	No[p]	Depends[q]	O(N)	Yes	O(N)[r]	No[s]	Yes	Yes	Scores[t]	Yes	Yes
Minimax	Yes	No	No	Yes[u]	No	No	No	No[b]	Spoilers	Yes	No[b]	No[b]	No	$O(N^2)$	Yes	$O(N^2)$	No[b][u]	No	No[b]	Ranking	Yes	Yes
Plurality/FPTP	Yes	No	No	No[b]	No	No[b]	No	No	Spoilers	Yes	Yes	Yes	No	O(N)	Yes	O(N)	N/A[v]	N/A[v]	No	Single mark	N/A	No
Score voting	No	No	No	No[b][c]	No	No[b]	Yes	Yes[d]	Yes	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes	Yes	Scores	Yes	Yes
Ranked pairs	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No[b]	Yes	Yes	No[b]	No[p][b]	Yes	$O(N^3)$	Yes	$O(N^2)$	No[b]	No	No[p][b]	Ranking	Yes	Yes
Runoff voting	Yes	Yes	No	No[b]	Yes	No[b]	No	No	Spoilers	No	No	No	No	O(N)[w]	Yes	O(N)[w]	Yes	Yes[x]	No	Single mark	N/A	No[y]
Schulze	Yes	Yes	Yes	Yes	Yes	Yes	No	No[b]	Yes	Yes	No[b]	No[p][b]	Yes	$O(N^3)$	Yes	$O(N^2)$	No[b]	No	No[p][b]	Ranking	Yes	Yes
STAR voting	No[z]	Yes	No[aa]	No[b][c]	Yes	No[b]	No	No	No	Yes	No	No	Depends[ab]	O(N)	Yes	$O(N^2)$	No	No	No[ac]	Scores	Yes	Yes
Sortition, arbitrary winner[ad]	No	No	No	No[b]	No	No[b]	Yes	Yes	No	Yes	Yes	Yes	Yes	O(1)	No	O(1)	Yes	Yes	Yes	None	N/A	N/A
Random ballot[ae]	No	No	No	No[b]	No	No[b]	Yes	Yes	Yes	Yes	Yes	Yes	Yes	O(N)	No	O(N)	Yes	Yes	Yes	Single mark	N/A	No

Voting with Ranked Ballots

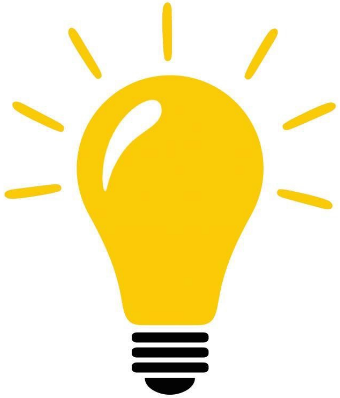


Utilitarian Voting with Ranked Ballots

[PR‘06]



Optimal Voting Rules with Ranked Ballots



Minimize distortion
(a.k.a. social welfare approximation
ratio in the worst case)

Notation

- N = set of n voters
- A = set of m alternatives

- $\vec{\sigma}$ = observed ranked preference profile
- $\vec{u}$ = underlying utility profile such that for each $i \in N$:
  - u_i is "consistent with" σ_i
  - u_i is "unit-sum": $\sum_a u_i(a) = 1$

- For $x \in \Delta(A)$:
  - $u_i(x) = \sum_a u_i(a) \cdot x(a)$
  - $sw(x, \vec{u}) = \sum_i u_i(x)$

Notation

- Distortion

$$\mathrm{dist}(x, \vec{\sigma}) = \sup_{\vec{u} \rhd \vec{\sigma}} \frac{\max_{a \in A} sw(a, \vec{u})}{sw(x, \vec{u})}$$

- Given voting rule f

$$dist(f) = \max_{\vec{\sigma}} \mathrm{dist}(f(\vec{\sigma}), \vec{\sigma})$$

- Instance-optimal rule f^*
  - Maps every preference profile $\vec{\sigma}$ to the instance-optimal solution $x^* \in \arg\min_x \mathrm{dist}(x, \vec{\sigma})$
  - Has the lowest distortion on each $\vec{\sigma}$, and therefore in the worst case over all $\vec{\sigma}$

Known Results

Deterministic Rules [CP'11; CNPS'17]

- The distortion of every deterministic voting rule is $\Omega(m^2)$
- The distortion of plurality is $O(m^2)$
- The instance-optimal rule can be computed in polynomial time

Randomized Rules [BCHLPS'15]

- The distortion of every randomized voting rule is $\Omega(\sqrt{m})$
- There exists a randomized voting rule with distortion $O(\sqrt{m} \cdot \log^* m)$
- The instance-optimal rule can be computed in polynomial time

Optimal Randomized Voting Rule

Theorem [EKPS'22]

❖ There exists an efficient randomized voting rule with distortion $O(\sqrt{m})$

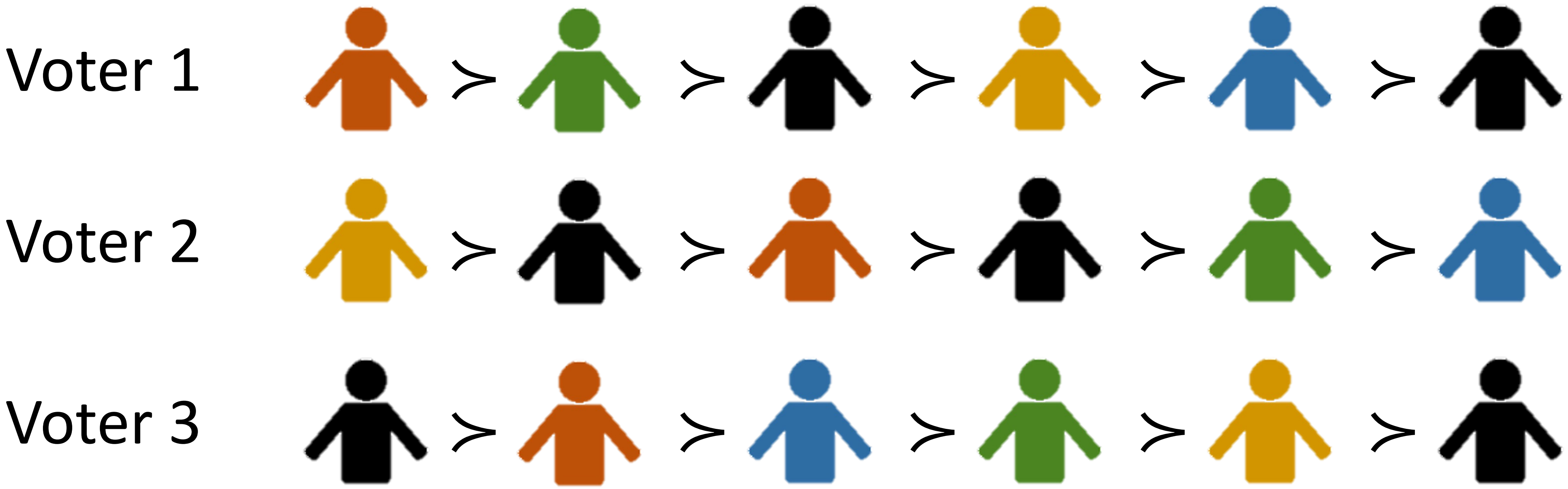
- "Nicer" rule
- Simpler proof in 3 steps:

  I. Define "stable lotteries"

  II. Prove the existence (and efficient computation) of stable lotteries via the minimax theorem

  III. Derive $O(\sqrt{m})$ distortion using stable lotteries

Step I: Define Stable Lotteries



- For a set of alternatives $S = \{$$\}$ and an alternative $a =$ 

$$V(a, S) = |\{i \in N : a \succ_i b, \forall b \in S\}| = 2$$

- Lottery $\mathcal{S}$ over sets of size k is stable if $\mathbb{E}_{S\sim\mathcal{S}}[V(a, S)] \leq {}^{n}/_{k}$ for every $a \in A$

Step II: Prove Stable Lotteries Exist

- Theorem: For every k, a stable lottery over committees of size k exists.
- Proof:
  - $\min_{\mathcal{S}} \max_{a \in A} \mathbb{E}_{S \sim \mathcal{S}}[V(a,S)] \leq \min_{\mathcal{S}} \max_{x \in \Delta(A)} \mathbb{E}_{S \sim \mathcal{S}, a \sim x}[V(a,S)]$
    $$= \max_{x \in \Delta(A)} \min_{\mathcal{S}} \mathbb{E}_{S \sim \mathcal{S}, a \sim x}[V(a,S)] \leq \frac{n}{k}$$
  - For any $x \in \Delta(A)$, consider the lottery $\mathcal{S}^*$, where we sample k alternatives i.i.d. according to x and replace any duplicates with arbitrary other alternatives
  - For each voter i:
    $$\Pr_{S \sim \mathcal{S}^*, a \sim x}[a \succ_i b, \forall b \in S] \leq \frac{1}{k+1}$$
  - Hence:
    $$\mathbb{E}_{S \sim \mathcal{S}^*, a \sim x}[V(a,S)] \leq \frac{n}{k+1} < \frac{n}{k} \quad \blacksquare$$

Step III: Ding Ding Ding!

Stable Lottery Rule

- W.p. ½ , find a stable lottery $\mathcal{S}$ over sets of size $\sqrt{m}$, sample $S \sim \mathcal{S}$, choose $a \in S$ uniformly at random
- W.p. ½, choose $a \in A$ uniformly at random

- **Theorem:** Stable lottery rule achieves $O(\sqrt{m})$ distortion.
  - Let a^* be an alternative maximizing social welfare
  - For any S: $sw(a^*, \vec{u}) \leq V(a^*, S) + \sum_{b \in S} sw(b, \vec{u})$
  - Taking expectation over $S \sim \mathcal{S}$:

$$sw(a^*, \vec{u}) \leq \mathbb{E}_{S\sim\mathcal{S}}[V(a^*, S)] + \mathbb{E}_{S\sim\mathcal{S}}[\textstyle\sum_{b \in S} sw(b, \vec{u})]$$

$$\leq 2\sqrt{m} \cdot \left(\frac{1}{2} \cdot \frac{n}{m} + \frac{1}{2} \cdot \mathbb{E}_{S\sim\mathcal{S}}\left[\frac{1}{|S|} \cdot \textstyle\sum_{b \in S} sw(b, \vec{u})\right]\right)$$

$$= 2\sqrt{m} \cdot sw(f(\vec{\sigma}), \vec{u}) \blacksquare$$

Thoughts

- Stable lotteries
  - Introduced by [CJMW19], who show the existence of a stronger form of stable lotteries which bounds $V(S', S)$ for all $S' \subseteq A$ instead of just $V(a, S)$ for all $a \in A$
  - Requires a much more intricate proof
- Stable committees
  - 16-stable committees exist [JMW20]: $V(a, S) \leq 16 \cdot \frac{n}{k}$ for all $a \in A$
  - Factor 16 cannot be improved to any lower than 2
  - Open question: Do 2-approximately stable committees exist?
- Novel connection
  - Choosing a single winner by choosing a random member of a suitably large fair committee
  - Connection between fairness (stable lotteries) and welfare (distortion)
    - “If you want to be efficient, it pays to be fair!”

Thoughts

- Lower bound
  - The lower bound [BCHL+15] can be improved to $\frac{\sqrt{m}}{2}$ with a tighter analysis
  - Open question: A gap of factor 4 between this lower bound and our $2\sqrt{m}$ upper bound
- Efficient computation
  - Minimax stable lottery value is at most $\frac{n}{k+1}$ whereas we only need $\frac{n}{k}$
  - Solve the two-player zero-sum game approximately via, e.g., multiplicative weights update
- Unit-range utilities
  - $\max_a u_i(a) = 1$ and $\min_a u_i(a) = 0$
  - Stable lottery rule continues to have $O(\sqrt{m})$ distortion for unit-range utilities
    - Distortion of harmonic rule [BCHL+15] increases from $\Theta\left(\sqrt{m \cdot \log m}\right)$ to $\widetilde{\Theta}\left(m^{2/3}\right)$

Robustness: Committee Selection

- What if our goal is to select a committee in the first place?
  - Need to define voter utilities for committees
    - Representation utilities: $u_i(S) = \max_{a \in S} u_i(a)$
    - Apriori, it is not clear if the best possible distortion increases or decreases with k

- Known results [CNPS'17]
  - Deterministic rules: $\Theta\left(1 + \frac{m \cdot (m-k)}{k}\right)$
  - Randomized rules:

2. ***Distortion, randomized rules:*** *There exists a randomized voting rule f^* such that*

$$\mathtt{dist}(f^*) \leqslant \begin{cases} 2\sqrt{m \cdot H_m} & \text{if } k \leqslant \frac{2 \cdot m \cdot H_m}{m + H_m}, \\ 4\sqrt{m \cdot k} & \text{if } \frac{2 \cdot m \cdot H_m}{m + H_m} < k \leqslant \left(\frac{m}{4}\right)^{\frac{1}{3}}, \\ \frac{m}{k} & \text{otherwise,} \end{cases}$$

where $H_m = \Theta(\log m)$ is the m^{th} harmonic number. Moreover, for every randomized voting rule f,

$$\mathtt{dist}(f) \geqslant \begin{cases} \frac{\sqrt{m}}{2} & \text{if } k \leqslant \frac{m \cdot (\sqrt{m} - 1)}{m - 1} \approx \sqrt{m}, \\ \frac{m}{k + m/k} & \text{otherwise.} \end{cases}$$

These bounds are tight up to a factor of $6.35 \cdot m^{1/6}$.

Robustness: Committee Selection

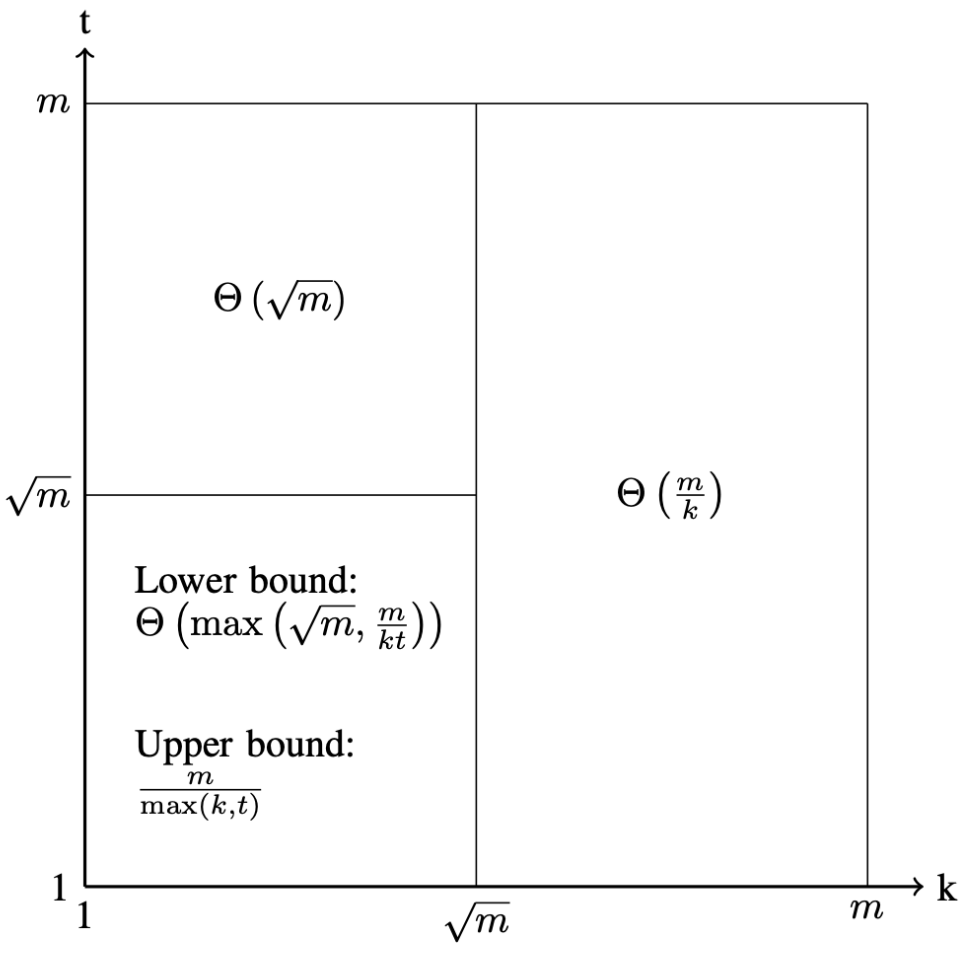
Stable Lottery Rule for Committees

- If $k \leq \sqrt{m}$:
  - W.p. ½ , find a stable lottery $\mathcal{S}$ over sets of size $k \cdot \sqrt{m}$, sample $S \sim \mathcal{S}$, and choose $S' \subseteq S$ of size $|S'| = k$ uniformly at random
  - W.p. ½, choose $S \subseteq A$ of size $|S| = k$ uniformly at random
- If $k \geq \sqrt{m}$
  - Choose $S \subseteq A$ of size $|S| = k$ uniformly at random

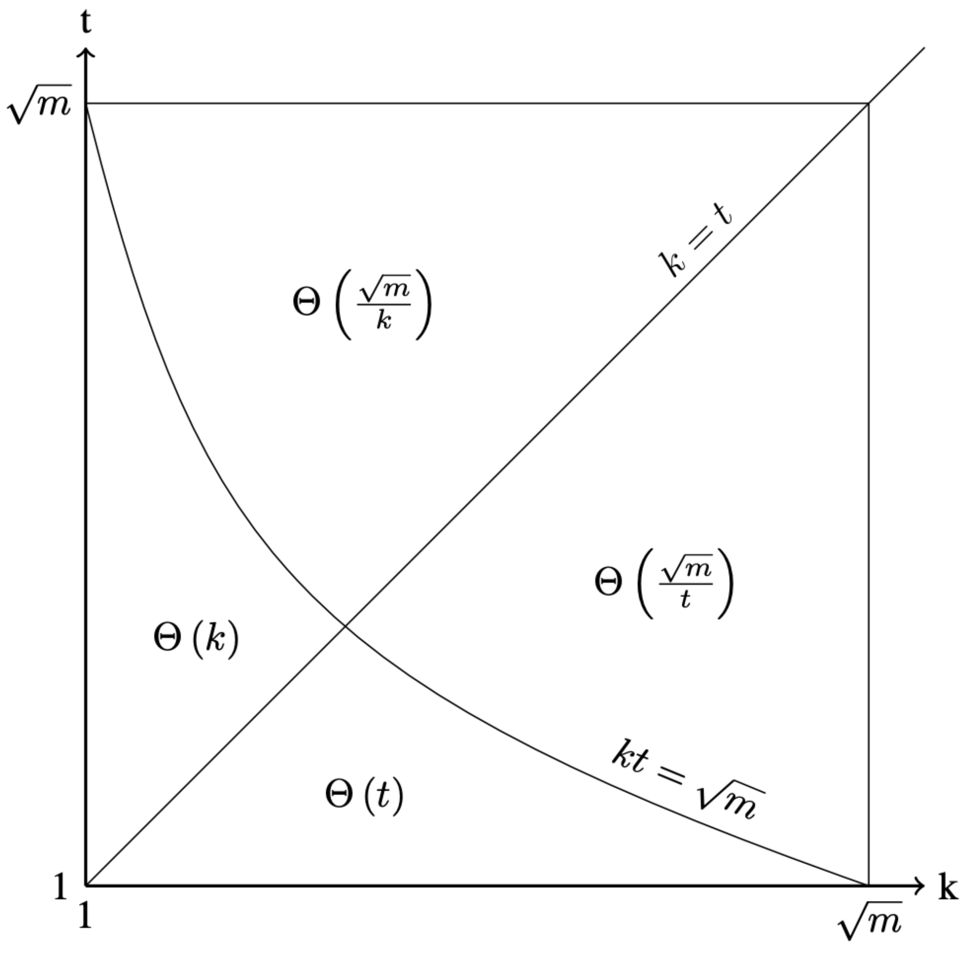
- Theorem [BHLS'22]:
  - Stable lottery rule for committees of size k achieves the optimal distortion of $\Theta\left(\min\left(\sqrt{m}, \frac{m}{k}\right)\right)$
- Corollary:
  - The best possible distortion (asymptotically) does not increase with k

Robustness: Partial Preferences

- What if each voter ranked only her top t alternatives?
  - Arbitrarily complete the preference profile and apply the stable lottery rule for committees!
- Theorem [BHLS‘22]:
  - Stable lottery rule for committees on top-t preferences has distortion $O\left(\min\left(\max\left(\sqrt{m}, \frac{m}{t}\right), \frac{m}{k}\right)\right)$
- Theorem [BHLS‘22]:
  - Every randomized voting rule on top-t preferences has distortion $\Omega\left(\min\left(\max\left(\sqrt{m}, \frac{m}{k \cdot t}\right), \frac{m}{k}\right)\right)$
- Corollary:
  - For $k = 1$ (single-winner), the bound is $\Theta\left(\max\left(\sqrt{m}, \frac{m}{t}\right)\right)$
  - Optimal $O(\sqrt{m})$ distortion is already achieved at $t = \sqrt{m}$
  - No benefit from asking voters to rank more than their top $\sqrt{m}$ alternatives!



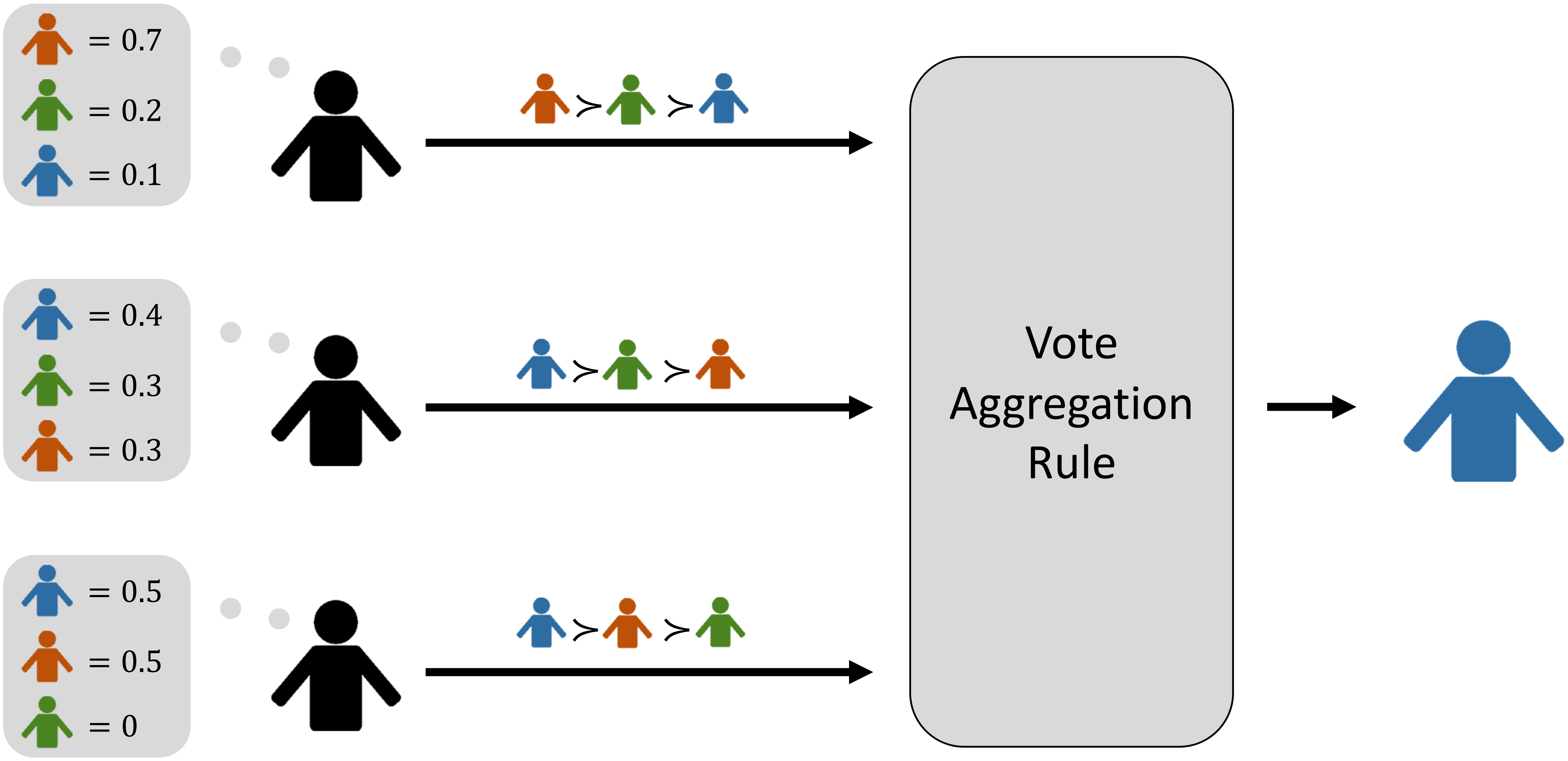
(a) Upper and lower bounds for different values of k and t.



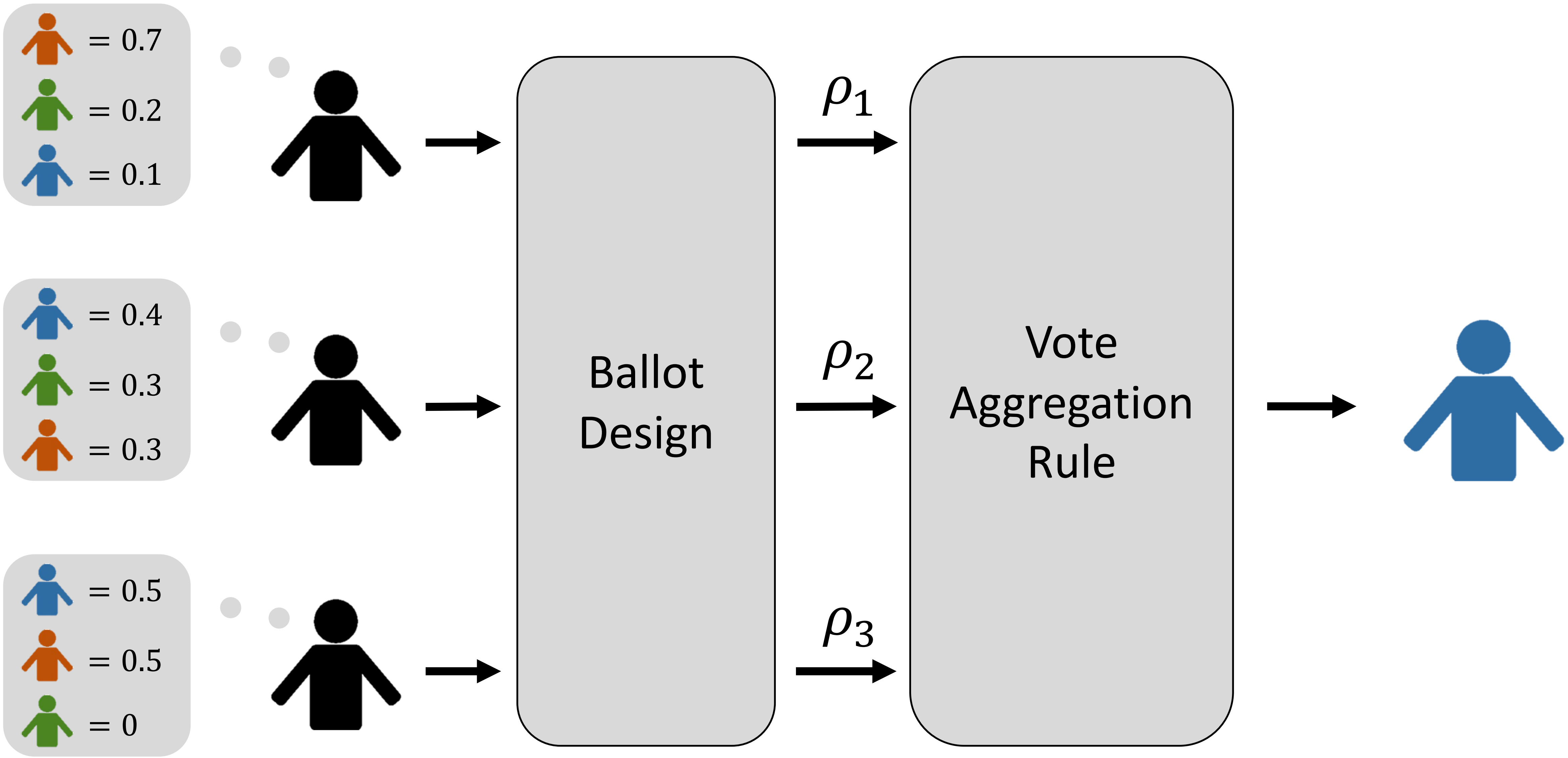
(b) Gap factor for $t, k \leqslant \sqrt{m}$.

- Open Question: Close the gap when $k = \omega(1)$ but $o(\sqrt{m})$ and $t = \omega(1)$ but $o(\sqrt{m})$

Utilitarian Voting with Ranked Ballots



Utilitarian Voting with Generic Ballots



Ballots

Ranked Ballot	1st	2nd	3rd	4th
A	○	●	○	○
B	●	○	○	○
C	○	○	○	●
D	○	○	●	○

Top-*t* Ballot	1st	2nd	3rd	4th
A	○	●	○	○
B	●	○	○	○
C	○	○	○	○
D	○	○	○	○

Range Voting	1 (Worst)	2	3	4 (Best)
A	○	○	●	○
B	○	○	●	○
C	●	○	○	○
D	○	●	○	○

Approval Ballot	1st
A	●
B	●
C	○
D	○

Optimal Voting with Optimal Ballot Design

- Tradeoff

Distortion

- Lowest distortion allowed by the ballot design when using its best aggregation rule

VS

Communication

- "Expressiveness" / "cognitive difficulty" imposed
- Crude measure: #bits communicated by each voter

How many bits of information does each voter need to communicate for us to achieve distortion d?

Optimal Voting with Optimal Ballot Design

- Single-winner voting ($k = 1$) [MPWS'19]

Ballot	Distortion	Communication
Ranked	$\Theta(\sqrt{m})$	$\Theta(m \cdot \log m)$
Optimal deterministic	Any d	$\tilde{\Theta}\left({}^{m}/_{d}\right)$
Optimal randomized	Any d	$\tilde{\Theta}\left({}^{m}/_{d^3}\right)$

- Committee selection of size k [MWS'20]

Ballot	Distortion	Communication
Ranked	$\Theta\left(\min\left(\sqrt{m}, {}^{m}/_{k}\right)\right)$	$\Theta(m \cdot \log m)$
Optimal deterministic	Any d	$\tilde{\Theta}\left({}^{m}/_{kd}\right)$
Optimal randomized	Any d	$\tilde{\Theta}\left({}^{m}/_{kd^3}\right)$

Quantitative Fairness

- If we knew the utility profile $\vec{u}$:
  - Efficiency would ask us to select $x^* \in \arg\max_x sw(x, \vec{u})$
  - What about fairness?

- Proportional Fairness

$$\mathrm{PF}(x, \vec{u}) = \sup_y \sum_i \frac{u_i(y)}{u_i(x)}$$

  - Maximum total % change in utilities when moving to any other distribution y
  - Folklore: If we knew $\vec{u}$, choosing $x^* \in \arg\max_x \prod_i u_i(x)$ would guarantee $\mathrm{PF}(x^*, \vec{u}) = 1$

Given only the preference profile $\vec{\sigma}$, how well can we minimize $\sup_{\vec{u} \rhd \vec{\sigma}} \mathrm{PF}(x, \vec{u})$?

Proportional Fairness

- Theorem [EKPS'22]:
  - Stable committee* rule with ranked ballots achieves $O(\sqrt{m})$ proportional fairness.

- Theorem [EKPS'22]:
  - There exists an efficient randomized voting rule with ranked ballots that achieves $O(\log m)$ proportional fairness.
    - Uses a stronger form of the minimax theorem with a slightly more intricate proof

- Theorem [EKPS'22]:
  - Every randomized voting rule with ranked ballots has proportional fairness $\Omega(\log m)$.

* Similar to the stable lottery rule

Proportional Fairness

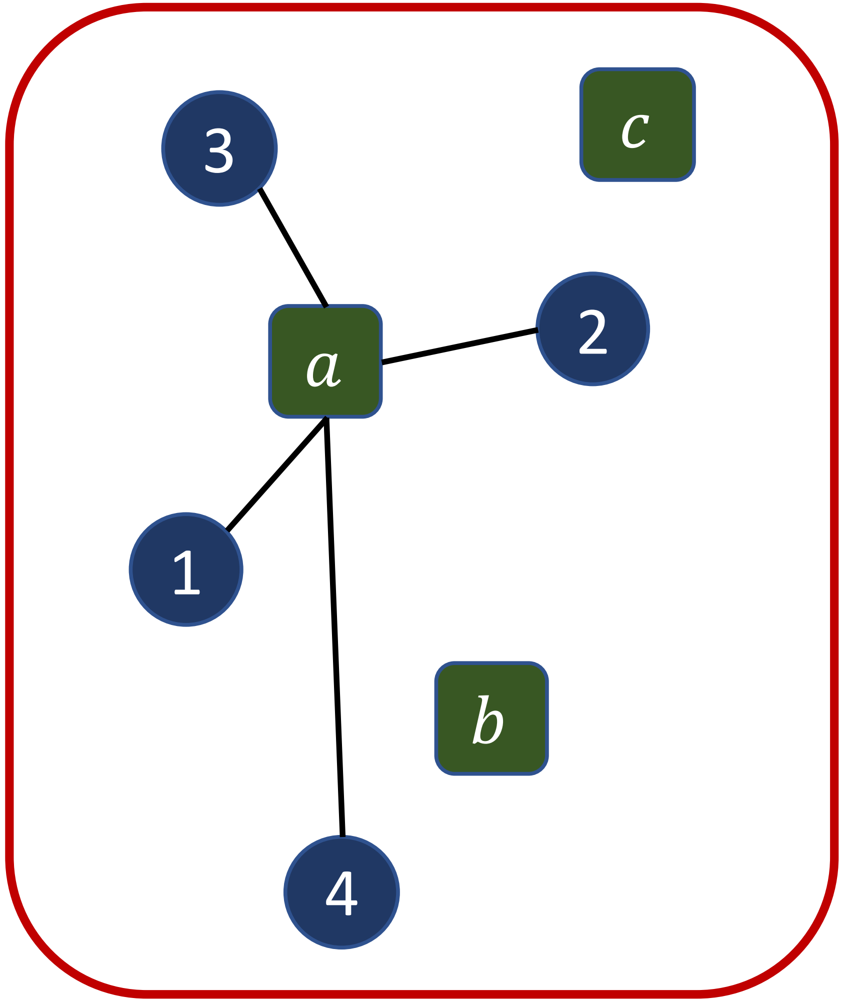
- Hmm, I don't really buy proportional fairness as a notion of fairness…
- Core
  - Definition: Any subgroup of x % of voters cannot find a Pareto improvement over the given outcome by allocating x % of the probability mass (or budget), for any x
  - Folklore: α proportional fairness implies α-approximation of the core
  - Corollary: There exists a distribution in $O(\log m)$-approximate core
  - Known lower bound: 2-approximate core
  - Open question: Close the gap!

Proportional Fairness

- Hmm, I don't really buy proportional fairness as a notion of fairness…

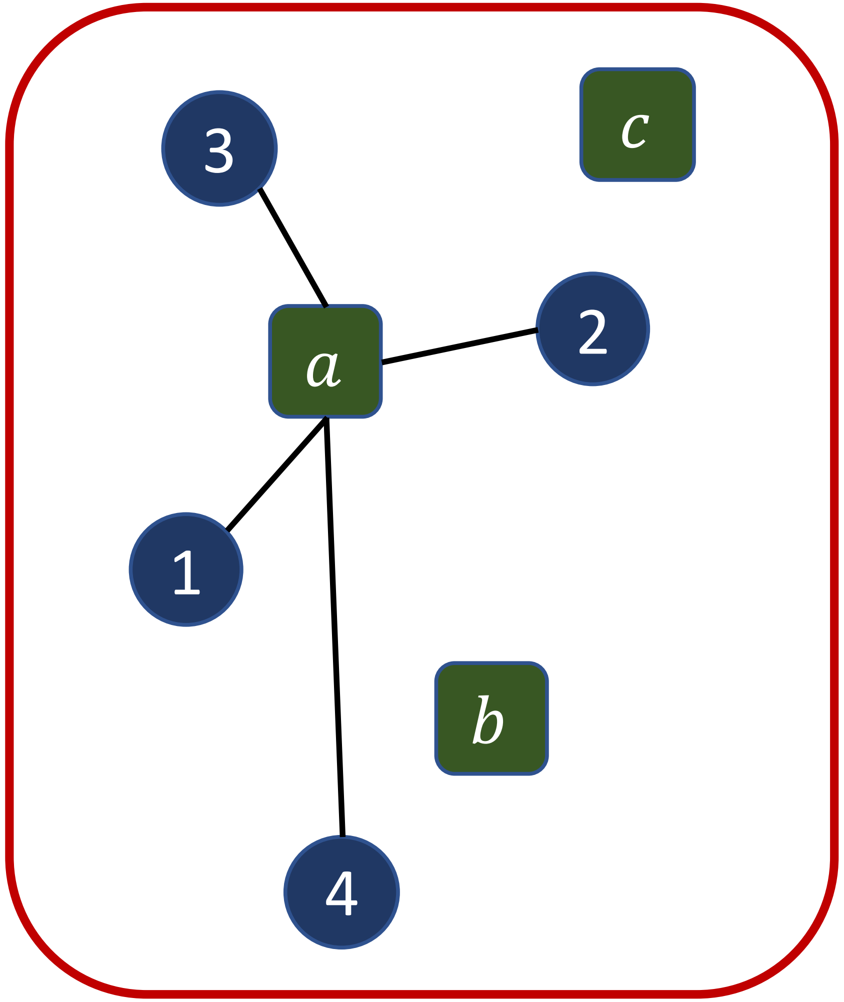
- Distortion with respect to the Nash welfare
  - Definition: $\mathrm{dist}^{\mathrm{Nash}}(x, \vec{\sigma}) = \sup_{\vec{u} \rhd \vec{\sigma}} \frac{\max_{a \in A} nsw(a, \vec{u})}{nsw(x, \vec{u})}$, where $nsw(x, \vec{u}) = \left(\prod_i u_i(x)\right)^{1/n}$
  - Folklore: α proportional fairness implies α distortion w.r.t. the Nash welfare
  - Corollary: There exists an efficiently computable distribution that gives $O(\log m)$ distortion with respect to the Nash welfare
  - Known lower bound: $e \approx 2.717$
  - Open question: Close the gap!

Future Work: Metric Distortion



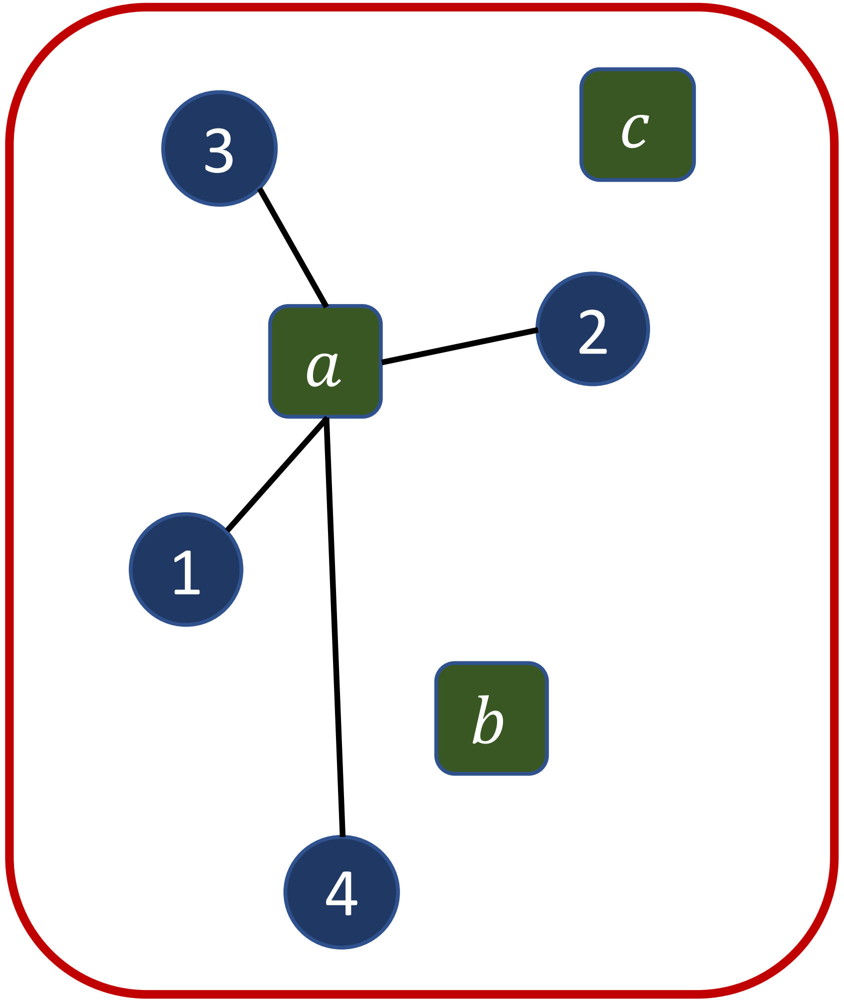
- Metric Distortion
  - Voters and candidates embedded in an underlying metric space (Δ inequality)
  - Cost of a voter for a candidate is the distance between the two
  - Approximation of social cost
- Ranked ballots, single-winner
  - Deterministic: 3 [GHS'20]
  - Randomized: in [2.0261, 3) [CR'21]
    - Open question!

Future Work: Metric Distortion



- Metric Distortion
  - Voters and candidates embedded in an underlying metric space (Δ inequality)
  - Cost of a voter for a candidate is the distance between the two
  - Approximation of social cost
- Ranked ballots, committee of size k
  - $c_i(S) = qt^h\text{–}\min_{a \in S} d(i, a)$
  - Trichotomy:
    - $1 \leq q \leq {}^{k}/_{3} : \infty$
    - ${}^{k}/_{3} < q \leq {}^{k}/_{2} : \Theta(n)$
    - ${}^{k}/_{2} < q \leq k : 3$
      - We only know how to achieve 9 in polynomial time

Future Work: Metric Distortion



- Metric Distortion
  - Voters and candidates embedded in an underlying metric space (Δ inequality)
  - Cost of a voter for a candidate is the distance between the two
  - Approximation of social cost
- What about other ballot formats?
- What about optimizing the ballot format?
- How can we model participatory budgeting in the metric distortion framework?

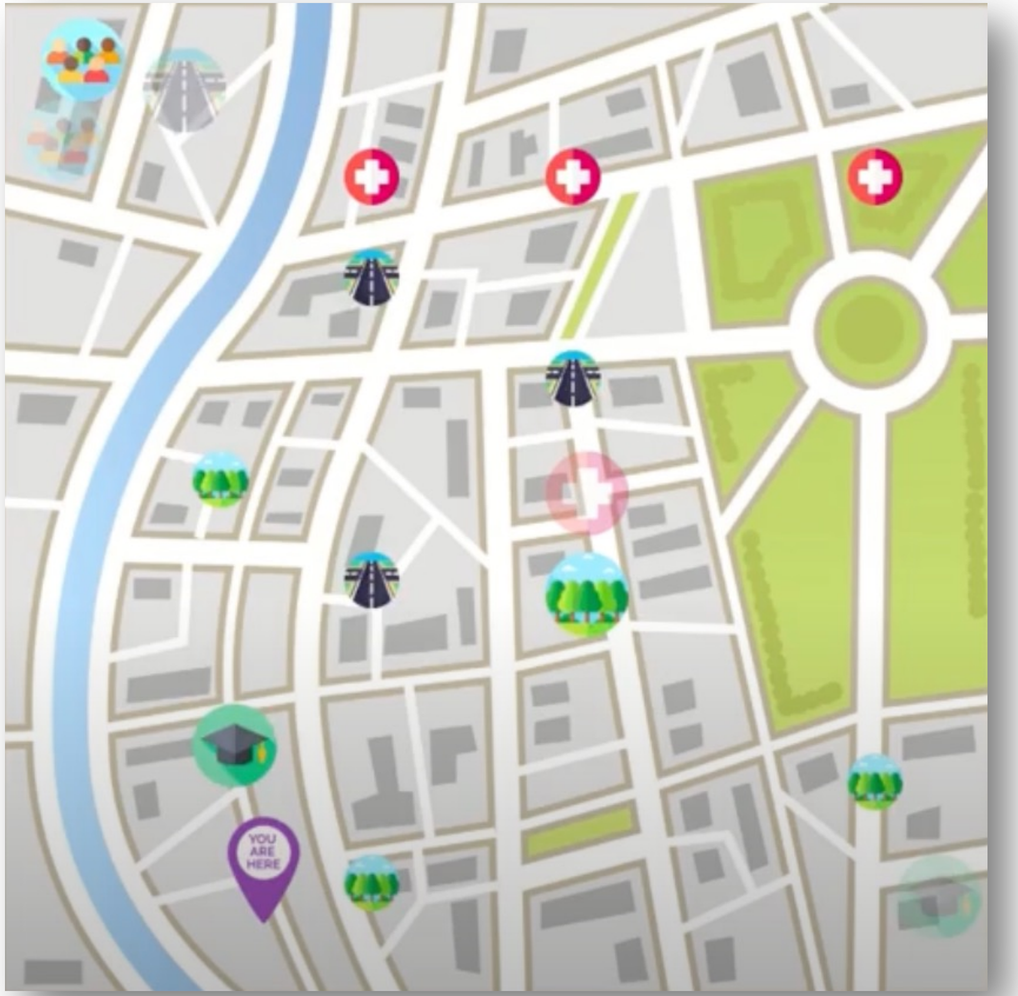
Future Work: Ballot Design

- Common ballot designs
  - Pairwise comparisons, “Do you like candidate a at least twice as much as candidate b?”, …
- Better models of cognitive burden
  - Psychology, HCI, …
- Voter errors in answering ballots
  - Expressive ballots can also induce errors
- Intangible aspects of ballot design
  - Barcelona PB team: “Knapsack votes are good because they help voters understand the limitations of the budget.”

Future Work: Distortion vs Other Desiderata

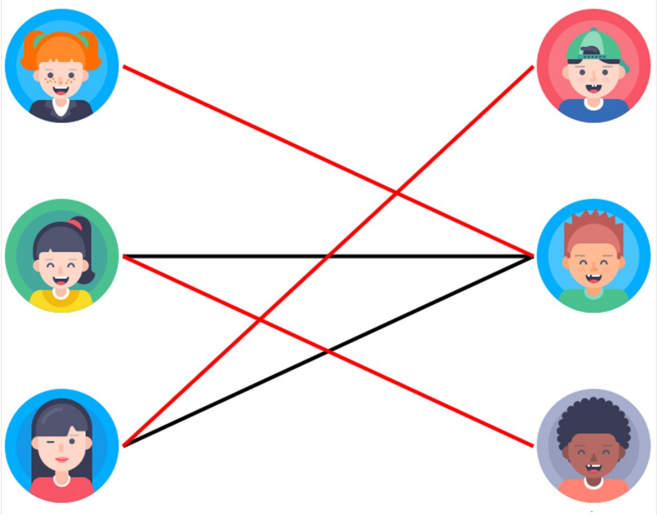
- Distortion & Truthfulness
  - With ranked ballots, optimal $\widetilde{\Theta}(\sqrt{m})$ distortion can be achieved via truthful aggregation
  - What happens with other ballot formats?
- Distortion & Axioms
  - Can we achieve low distortion together with popular axioms?
- Distortion & Explainability
  - Explaining the voting rule vs explaining what it does

Future Work: More Complex Voting Paradigms



- Design optimal voting rules for more complex voting paradigms
  - Participatory budgeting
  - Districting
- Model end-to-end voting
  - In participatory budgeting, voting is but the final step of a year-long process
- Compare different models of democracy
  - E.g., direct democracy, representative democracy, and liquid democracy
  - [Borodin, Lev, S., Strangway, 2019]: Compared primary vs direct elections

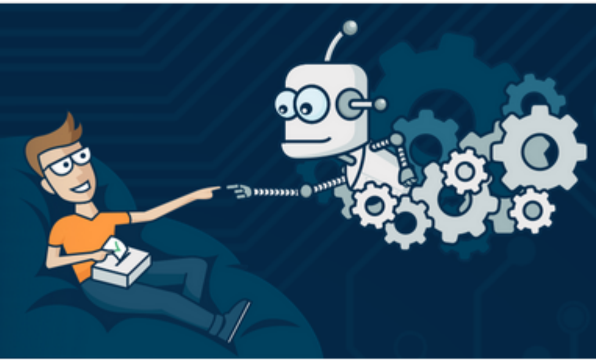
Future Work: Distortion Beyond Voting



- Distortion (“price of missing information”) beyond voting
  - Fair division [HS‘21; EFS’22]
  - One-sided Matching [ABFV’22; MML’21]
  - Two-sided matching?
  - Coalition formation in hedonic games?
  - Graph problems [AS’16; AZ’17; AZ’18; AA‘18]

AI-Driven Decisions

RoboVote is a free service that helps users combine their preferences or opinions into optimal decisions. To do so, RoboVote employs state-of-the-art voting methods developed in artificial intelligence research.
Learn More



Poll Types

RoboVote offers two types of polls, which are tailored to different scenarios; it is up to users to indicate to RoboVote which scenario best fits the problem at hand.

Objective Opinions

In this scenario, some alternatives are objectively better than others, and the opinion of a participant reflects an attempt to estimate the correct order. RoboVote's proposed outcome is guaranteed to be as close as possible — based on the available information — to the best outcome. Examples include deciding which product prototype to develop, or which company to invest in, based on a metric such as projected revenue or market share.

Subjective Preferences

In this scenario participants' preferences reflect their subjective taste; RoboVote proposes an outcome that mathematically makes participants as happy as possible overall. Common examples include deciding which restaurant or movie to go to as a group.

Ready to get started?

CREATE A POLL

Thank you!

Questions?